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Editorial

After three years of waiting, the faithful readers of QARCH will be delighted to receive this new issue. It seems that an unknown number of past issues are irretrievably lost, so we start with CXII. On succeeding Leo Lai as Editor, we would like to thank him and his deputy, Kaan Cem Ketenci, for their hard work in editing QARCH.

We are aiming to publish at least one issue every year, and ask for our readers' continued interest. Please do write to us at archim-qarch@srcf.net with your progress on these problems and with any problems that you would like to see printed in the future. We look forward to hearing some creative solutions.

New Questions

1. Does every polynomial $f(X, Y)$ in two variables that is bounded below achieve its infimum?
2. Start with some checkers on an $n \times n$ chess board. You are allowed to add a checker in a square if at least two of its neighbours have checkers in them. What is the smallest number of checkers that you can start with and still fill in the whole board?
3. The five sides and five diagonals of a regular pentagon are drawn on a piece of paper. Two people play a game, in which they take turns to colour one of these ten line segments. The first player colours line segments blue, while the second player colours line segments red. A player cannot colour a line segment that has already been coloured. A player wins if they are the first to create a triangle in their own colour, whose three vertices are also vertices of the regular pentagon. The game is declared a draw if all ten line segments have been coloured without a player winning. Determine whether the first player, the second player, or neither player can force a win.
4. There is a group of people in a room with pencils and some paper. Each person knows a secret number. They wish to compute the average of all their numbers without revealing any other information to any member of the group. Is this possible?

5. Let G be a graph embedded on the sphere such that all faces are triangles. Show that the vertices can be three-coloured if and only if G has an Euler cycle.
6. What are the functions $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f(z)$ and $f(1/z)$ are meromorphic?
7. Do there exist two non-homeomorphic sets $X, Y \subseteq \mathbb{R}^2$ such that $X \times [0, 1]$ and $Y \times [0, 1]$ are homeomorphic?
8. Let p be a 7 mod 8 prime. Show that

$$\sum_{a=1}^{\frac{p-1}{2}} a \left(\frac{a}{p} \right) = 0,$$

where $\left(\frac{a}{p} \right)$ is the Legendre symbol.

9. Is $\mathbb{Z}[2i]$ a UFD? What about $\mathbb{Z}[i/2]$? Can $\mathbb{Z}[\sqrt{-5}]$ be made into a UFD by inverting one element (i.e. is there some element $\alpha \in \mathbb{Z}[\sqrt{-5}]$ such that $\mathbb{Z}[\sqrt{-5}, \alpha^{-1}]$ is a UFD)?
10. Does every subset S of \mathbb{R} with $|S| = |\mathbb{R}|$ have a subset that is order-isomorphic to \mathbb{R} ?
11. The Möbius maps send all circles and lines to either a circle or a line. Are there any other such functions on the complex plane? Find all of them.
12. Is F_2 , the free group on two generators, isomorphic to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$?
13. What are the conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$?
14. For what real numbers α do integer powers of α get arbitrarily close to some integer?

Solutions and Comments

1. With no progress, but still considerable interest being shown, we ask further, (16.) Do there exist two matrices that are conjugate in $SL_2(\mathbb{Z}/n\mathbb{Z})$ for all natural numbers n but are not conjugate in $SL_2(\mathbb{Z})$?

Problems, solutions, partial solutions, and comments to:
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